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ONE-DIMENSIONAL ADVECTIVE–DISPERSIVE TRANSPORT INTO A DEEP LAYER HAVING A VARIABLE SURFACE CONCENTRATION

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SUMMARY

Two analytic closed form solutions for the one-dimensional advection–dispersion equation are presented. These solutions take account of changes in surface concentration with time as mass is transported into the soil. The first solution is developed for the case of rapid landfill construction. The second solution considers a time-dependent mass input to the landfill. The use of these solutions is illustrated by a number of examples and it is shown that consideration of the finite mass of contaminant within the landfill can significantly affect the concentration profiles beneath the landfill.

INTRODUCTION

The evaluation and design of landfills and other waste disposal sites generally requires a consideration of the potential increase in the concentration of dissolved contaminants within the groundwater beneath the landfill.

A traditional approach to the prediction of concentration profiles is to solve the advection–dispersion equation (see, for example, References 1–3). Finite element and finite difference techniques have been extensively used for this purpose (see, for example, Reference 4). Although such methods give reasonable results, the computational effort involved is quite large, particularly if it proves necessary to calculate concentration profiles at large times. Special care is also required to avoid numerical inaccuracies. More recently, Rowe and Booker[5–7] have developed a semi-analytic technique, applicable to a wide range of practical boundary conditions for one-, two- and three-dimensional problems which avoids these difficulties.

Although these techniques reduce the computational effort necessary to obtain the concentration distribution, like finite element and finite difference approaches, they also require access to a computer and some knowledge of numerical analysis. For complex problems, there

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is, of course, no alternative to adopting one of these computer methods. However, for simpler problems, analytic solutions which can be used without a computer or knowledge of numerical analysis are to be preferred.

In many practical situations, the mass transport of contaminant is essentially one dimensional and the deposit is relatively homogeneous and sufficiently deep that over the time range of interest, the deposit may be considered to be a halfspace. In such cases, it may be possible to find an analytic solution to the governing equations. For example, Ogata[8] developed an analytic solution for the one-dimensional advective transport in a halfspace with a surface concentration which did not vary with time. This result has been extensively used in the interpretation of field and laboratory data. Numerous other solutions have been developed. For example, Lindstrom and Narasimhan[9] solved the initial value problem of mass transport of a trace concentration of a previously distributed chemical in a water-saturated porous medium, while Lindstrom[10] presented a mathematical model for the pulsed dispersion of a trace chemical in a water-saturated sorbing porous medium. These solutions have many practical applications; however, they do not cover the situation where the source concentration varies with time due to the placing of a landfill over a finite period of time or, more importantly, due to mass transport of contaminant from a landfill into the underlying soil.

This paper develops an analytic closed form solution for the advection- dispersion equation which accounts for such factors. In particular, solutions are developed for a homogeneous halfspace where (a) the surface concentration varies with time due to mass transport into the soil, and (b) the surface concentration varies because the mass input into the landfill varies linearly with time, while at the same time there is mass transport into the deposit.

The solutions are presented in a form which may readily be used by practising engineers and groundwater hydrologists and the important effects and practical implications are illustrated. The closed form solution may also be useful for validating finite element and finite difference codes and associated numerical procedures.

GOVERNING EQUATIONS

For one-dimensional Fickian transport of a single contaminant species, the mass flux f in the z direction is given by

$$f = nvc - nD\frac{\partial c}{\partial z} \tag{1}$$

where c is the increase in concentration above the equilibrium state, n is the soil porosity, D is the coefficient of hydrodynamic dispersion, and v is the seepage velocity.

Assuming that there is equilibrium controlled ion exchange and that the concentration of one of the exchange ions is relatively low, the differential equation governing mass transport of contaminant (including the effect of linear sorption) is given by

$$\frac{\partial f}{\partial z} + (n + \rho K)\frac{\partial c}{\partial t} = 0 \tag{2}$$

where ρ is the bulk density of solid and K is a distribution coefficient which may be determined experimentally for a given soil and solute species (assuming linear adsorption).

Equations (1), (2) can be solved by introducing a Laplace transform

$$\bar{c} = \int_0^\infty e^{-st} c(z,t)\, dt \tag{3}$$

It is then found that

$$D\frac{\partial^2 \bar{c}}{\partial z^2} - v\frac{\partial \bar{c}}{\partial z} - \left(1 + \frac{\rho K}{n}\right)s\bar{c} = 0 \tag{4}$$

It is convenient now to introduce the quantities

$$\gamma = \sqrt{\left(\frac{1 + \rho K/n}{D}\right)} \tag{5a}$$

$$a = \frac{v}{2\gamma D} \tag{5b}$$

then it is readily shown that equation (4) has the solution

$$\bar{c} = X\,e^{\lambda z} + Y\,e^{\mu z} \tag{6}$$

where

$$\lambda = \gamma(a - \xi)$$

$$\mu = \gamma(a + \xi)$$

and

$$\xi = \sqrt{(s + a^2)} \quad \text{(branch with a positive real part)}$$

and X, Y are to be determined from the boundary conditions.

For problems involving the halfspace ($z > 0$), it is clear that the concentration must remain bounded for large values of z and so the coefficient Y must vanish. This means that equation (6) may be written

$$\bar{c} = \bar{c}_T\,e^{\lambda z} \tag{7}$$

where $\bar{c}_T$ denotes the concentration at the surface of the halfspace.

It is also possible to find a relation between the flux f_T entering the halfspace and the surface concentration from equations (1), (7) and it is found that:

$$\bar{f}_T = nD\mu\bar{c}_T \tag{8}$$

ADVECTIVE DISPERSIVE TRANSPORT INTO A HOMOGENEOUS HALFSPACE

The theory developed in the previous section will now be used to develop the solution for advective–dispersive transport into a homogeneous halfspace for the problem shown in Figure 1.

Landfills are of finite extent and have a limited active life. Typically, landfills are constructed in cells. Decomposition will commence immediately after construction of a particular cell, and for a period of time (which will depend upon the specific conditions) the concentration of a particular contaminant in the leachate will increase until a maximum is reached. This process may take several years and can be modelled, using superposition, if the details are known. However, if the landfill is constructed on a clay deposit or clay liner, the time to reach peak concentration is often small compared with the time scale imposed by the slow pollutant migration through the clay.

If the initial distribution of contaminant within the landfill is known, then the subsequent spatial and temporal variation in concentration of contaminant both within the landfill and the underlying soil could be modelled by treating the landfill and soil as different layers having

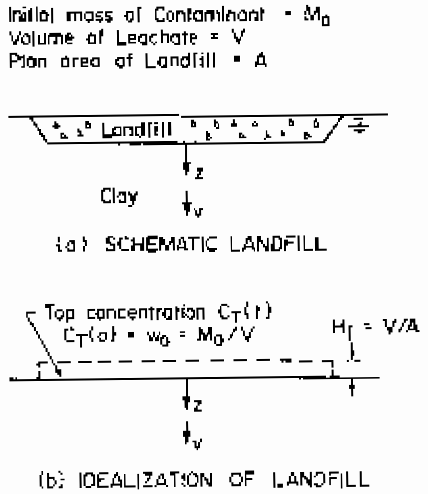

Figure 1. Problem idealization

appropriate properties. This situation may be analysed using the semi-analytic technique proposed by Rowe and Booker.[7] However, in many practical situations, the initial distribution of contaminant in the landfill will be heterogeneous and it will not be possible to precisely define this distribution of contaminant because of the random nature of its placement together with subsequent indeterminant advective–dispersive mixing which occurs in many landfills. In these cases, engineering judgement is required to assess reasonable bounds on the likely mass of the contaminant in the landfill and in view of this uncertainty a sophisticated numerical analysis may not be appropriate, particularly in preliminary design. In these cases it would be desirable to have a simple closed form solution which could be readily used in preliminary sensitivity studies.

A closed form analytic solution which gives the spatial and temporal variation in contaminant concentrate within the soil beneath a landfill can be derived provided that we are only concerned with the average concentration of contaminant within the landfill itself at any particular time. This type of model, which ignores the spatial variation of concentration within a reservoir (in this case the landfill), has been termed a lumped-parameter model[11] and has found good practical application in other areas of water research (see, for example, Reference 12). The adoption of a lumped-parameter model is equivalent to considering the limit as the diffusion–dispersion coefficient within the reservoir (landfill) tends to infinity. This gives rise to a concentration gradient within the landfill which tends to zero. It should be noted, however, that the product of the infinite coefficient of dispersion–diffusion and zero concentration gradient within the landfill gives rise to a finite non-zero limit and hence there will be both diffusive–dispersive and advective transport from the landfill into the underlying clay. Thus continuity of flow, flux and concentration at the landfill–soil interface ($z = 0$) gives:

$$n^+ v^+ c = nvc \tag{9a}$$

$$n^+ v^+ c - n^+ D^+ \left(\frac{\partial c}{\partial z}\right)\bigg|_{z=-\varepsilon} = nvc - nD\left(\frac{\partial c}{\partial z}\right)\bigg|_{z=\varepsilon} = f_T(t) \tag{9b}$$

where $|\varepsilon| \ll 1$, and the superscript + denotes properties of the landfill, and the subscript T denotes the flux at the landfill–soil interface ($z = 0$).

Considering this situation, suppose that at time $t = 0$, a mass M_0 of the contaminant species of interest overlies a surface area A of a deep homogeneous layer and that no further increase in mass of contaminant occurs after this time. For the sake of simplicity, it will also be assumed that ground water equilibrium has been reached in the landfill so that the time-average volume of leachate V remains essentially constant. Under these circumstances, a consideration of mass balance shows that

$$c_T(t)V = M_0 - A\int_0^t f_T(\tau)\,d\tau \tag{10a}$$

where the subscript T indicates, as before, the surface value of the variable. Equation (10) simply says that the mass of contaminant in the landfill at time t is equal to the initial mass of contaminant less the mass transported into the soil between time zero and time t.

Equation (10a) may be written in the alternative form:

$$c_T(t) = \omega_0 - \frac{1}{H_f}\int_0^t f_T(\tau)\,d\tau \tag{10b}$$

where $H_f = V/A$ is the equivalent height of contaminant, and $\omega_0 = M_0/V$ is the initial concentration of contaminant in the landfill. We notice in passing that the surface concentration will remain constant when $H_f \to \infty$.

Taking a Laplace transform of equation (10b) gives

$$\bar{c}_T = \frac{\omega_0}{s} - \frac{\bar{f}_T}{sH_f} \tag{11}$$

This can be rewritten as

$$\bar{c}_T = \frac{\omega_0}{a-b}\left(\frac{a}{\xi(\xi+a)} - \frac{b}{\xi(\xi+b)}\right) \tag{12}$$

in which

$$b = \frac{nD_T}{H_f} - a$$

Inversion of the Laplace transform[13] in (12) then gives the time-dependent variation in surface concentration, viz.

$$c_T(t) = \frac{\omega_0}{a-b}\,e^{-a^2 t}\left[a\phi(a\sqrt{t}) - b\phi(b\sqrt{t})\right] \tag{13a}$$

where

$$\phi(p\sqrt{t}) = e^{p^2 t}\,\mathrm{erfc}(p\sqrt{t}) \quad (p = a \text{ or } b) \tag{13b}$$

The variation in concentration with depth and time can be obtained by combining (7), (8) and (11) to give

$$\bar{c} = \frac{\omega_0}{s + \dfrac{nD\mu}{H_f}}\,e^{\lambda z} \tag{14a}$$

which can then be expressed in the form

$$\bar{c} = \frac{\omega_0 e^{a\kappa}}{(a-b)}\left[\frac{a\,e^{-\kappa\xi}}{\xi(\xi+a)} - \frac{b\,e^{-\kappa\xi}}{\xi(\xi+b)}\right] \tag{14b}$$

where $\kappa = \gamma z$.

Inverting (14b) then gives the complete solution, viz.

$$c(z,t) = \frac{\omega_0\,e^{\mu\kappa - a^2 t}}{a-b}[af(a,t) - bf(b,t)] \tag{15a}$$

where

$$f(q,t) = e^{q\kappa + q^2 t}\,\mathrm{erfc}\left(q\sqrt{t} + \frac{\kappa}{2\sqrt{t}}\right) \quad (q = a \text{ or } b) \tag{15b}$$

If we allow $H_f \to \infty$ so that the surface concentration is constant, we find that $b = -a$ and equation (15) reduces to the Ogata[8] solution.

ADVECTIVE DISPERSIVE TRANSPORT INTO A HOMOGENEOUS HALFSPACE WITH VARIABLE MASS INPUT

Suppose now that the mass input onto the landfill is variable, so that after an elapsed time t mass $M(t)$ has been dumped. It is now easy to establish that

$$c_T(t) = \omega(t) - \frac{1}{H_f}\int_0^t f_T(\tau)\,d\tau \tag{16}$$

where $\omega(t) = M(t)/V$ is the surface concentration if there is no mass transport into the soil. The solution parallels that given in the previous section and it is found without difficulty that

$$\bar{c} = \frac{s\bar{\omega}\,e^{\lambda z}}{s + \dfrac{\mu n D}{H_f}} \tag{17}$$

We thus see that using the convolution theorem, or equivalently, the Boltzmann superposition principle

$$c(z,t) = \omega(0)J(t) + \int_0^t J(t-\tau)\frac{d\omega(\tau)}{d\tau}\,d\tau \tag{18}$$

where $J(t)$ is given by equation (15a), viz.

$$J(t) = \frac{e^{a\kappa - a^2 t}}{a-b}[af(a,t) - bf(b,t)]$$

Equation (18) can be used to evaluate the concentration profile for any continuous variation of mass input with respect to time.

Sometimes it is more convenient to use equation (17) directly. Thus, if the mass input is proportional to the elapsed time t, so that $\omega(t) = mt$, then

$$\bar{\omega} = m/s^2$$

$$\bar{c} = \frac{m\,e^{\lambda z}}{s\left(s + \dfrac{\mu n D}{H_f}\right)} \tag{19a}$$

Equation (19) can be rewritten as

$$\bar{c} = m\,e^{\alpha\kappa}\left[\frac{A\,e^{-\kappa\xi}}{\xi(\xi - a)} + \frac{B\,e^{-\kappa\xi}}{\xi(\xi + a)} + \frac{C\,e^{-\kappa\xi}}{\xi(\xi + a)^2} + \frac{D\,e^{-\kappa\xi}}{\xi(\xi + b)^2}\right] \tag{19b}$$

where

$$A = \frac{1}{4a(b+a)}$$

$$B = \frac{-(a+b)}{4a(b-a)^2}$$

$$C = \frac{1}{2(b-a)}$$

$$D = \frac{b}{(b+a)(b-a)^2}$$

Inverting (19b) then gives:

$$c(z,t) = m\chi(t) \tag{20a}$$

$$\chi(t) = m\,e^{u\kappa - a^2 t}\left[Af(-a,t) + Bf(a,t) - Cf'(a,t) + Df(b,t)\right] \tag{20b}$$

where the function $f(q,t)$ was defined by (15b) and

$$f'(q,t) = \frac{\partial}{\partial p} f(q,t)$$

$$= (\kappa + 2qt)f(q,t) - 2\sqrt{(t/\pi)}\,e^{-\kappa^2/4t} \tag{20c}$$

APPLICATION

The concentration at any depth z and time t can be determined from (15) for instantaneous deposition of mass in the landfill and from (20) for the case where mass input is proportional to time. These equations can be readily evaluated using a hand calculator, observing that

$$f(q,t) = e^{-\kappa^2/4t}\phi(X)$$

where

$$X = \left(q\sqrt{t} + \frac{\kappa}{2\sqrt{t}}\right)$$

Values of $\phi(X)$ are given graphically in Figure 2 for $\chi \geqslant 0$. Alternatively, for $X \geqslant 0$, $\phi(X) \simeq a_1 w + a_2 w^2 + a_3 w^3 + a_4 w^4 + a_5 w^5$, where $w = 1/(1 + rX)$ and the coefficients r, $a_1 \cdots a_5$ are given in Reference 13 & section 7.9.26, p. 299. Solutions can be obtained for negative arguments by noting that

$$\phi(-x) = 2\,e^{x^2} - \phi(x)$$

Illustrative example

To illustrate the application of the theoretical solutions given by (13) and (15) consider a hypothetical landfill constructed on a deep clay layer. Assuming that the height of leachate (i.e. volume of leachate per unit area) $H_f = 1$ m, the clay porosity is 0·4, the downward seepage velocity $v = 0{\cdot}005$ m/a, the dispersion/diffusion coefficient is $D = 0{\cdot}01$ m^2/a for the species of interest, and sorption of the species is given by $\rho K = 1{\cdot}2$. From this data, the key parameters $\gamma = 20$; $a = 0{\cdot}0125$ and $b = 0{\cdot}0675$ can be calculated. Thus, the surface concentration 100 years after construction of the landfill may be calculated from (15), viz.

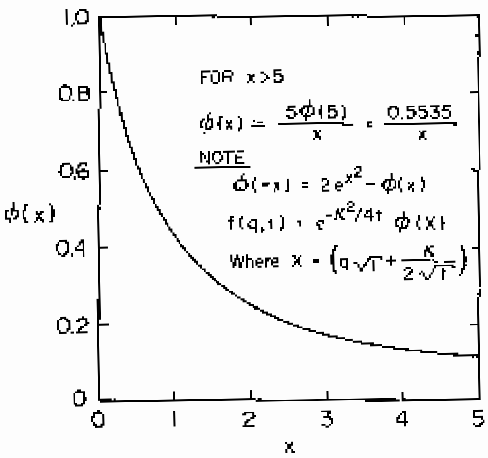

Figure 2. Function ϕ used to evaluate equations (15) and (20)

$$\frac{c_T}{\omega_0} = \frac{e^{-n^2 t}}{a-b}[a\phi(a\sqrt{t}) - b\phi(b\sqrt{t})]$$

where ω_0 is the initial surface concentration. Evaluating $\phi(a\sqrt{t})$ and $\phi(b\sqrt{t})$ from Figure 2 then gives

$$\therefore \frac{c_T}{\omega_0} = \frac{e^{-0{\cdot}0125t} \times 100}{0{\cdot}125 - 0{\cdot}675}[0{\cdot}125 \times 0{\cdot}87 - 0{\cdot}675 \times 0{\cdot}535] = 0{\cdot}45$$

The concentration at any depth z may be calculated from (15). For example, the concentration at a depth of 2 m after 100 years may be calculated using the values of γ, a, b, deduced above for $t = 100$ years and $\kappa = \gamma z = 40$. Thus, at $z = 2$ m, $t = 100$ years

$$\frac{c(t)}{\omega_0} = \frac{e^{a\kappa - a^2 t}}{a-b}[af(a,t) - bf(b,t)]$$

where $f(a,t)$ can be determined from Figure 2 as follows.

Let $X = a\sqrt{t} + \kappa/(2\sqrt{t}) = 2{\cdot}125$

$$\therefore f(a,t) = e^{-\kappa^2/4t}\phi(X) = e^{-4} \times \phi(2{\cdot}125) = 0{\cdot}0183 \times 0{\cdot}243 = 0{\cdot}00445$$

A similar calculation may be performed for $f(b,t)$ yielding $c(t = 100)/\omega_0 = 0{\cdot}0056$. Thus after 100 years, the average concentration of contaminant in the landfill has reduced to 45 per cent of the initial value, while the concentration at 2 m is only 0·56 per cent of the original surface value ω_0.

RESULTS

For cases in which the surface concentration remains constant, the effect of sorption can be directly combined with the effects of diffusion by considering a lumped diffusion coefficient $D^* = D/(1 + \rho K/n)$ (see, for example, Reference 3). However, this simplification ceases to be valid once the boundary conditions are flux controlled. This has been demonstrated numerically by Rowe *et al.*[14] and is apparent from the analytic solutions developed in the previous section. For example, taking the seepage velocity $v = 0$ (to simplify the expressions), equation (15) can be rewritten as

$$c(z,t) = \omega_0 \exp\left(\frac{nDz}{H_f} + n^2\frac{(1 + \rho K/n)Dt}{H_f^2}\right)\operatorname{erfc}\left(\frac{n}{H_f}\sqrt{[(1 + \rho K/n)Dt]} + \frac{z}{2}\sqrt{\frac{1 + \rho K/n}{Dt}}\right) \quad (21a)$$

or expressed in terms of the lumped parameter D^*

$$c(z,t) = \omega_0 \exp\left(\frac{nzD^*(1+\rho K/n)}{H_f} + \frac{n^2(1+\rho K/n)^2 D^* t}{H_f^2}\right)$$
$$\times \operatorname{erfc}\left(\frac{n}{H_f}\sqrt{[(1+\rho K/n)^2 D^* t]} + \frac{z}{2\sqrt{(D^* t)}}\right) \tag{21b}$$

An inspection of equation (21) shows that although the term D^* does appear, it will not in general provide a good normalizing parameter since for a given value of D^*, the solution also depends on the magnitude of the term $(1 + \rho K/n)$ and hence on the relative magnitudes of the parameters D, ρK and n used to obtain D^*.

The lumped parameter D^* can be readily determined in the laboratory by performing a diffusion test in which the source concentration remains constant (i.e. $H_f \to \infty$), since in this case (21) reduces to

$$c(z,t) = \omega_0 \operatorname{erfc}\left(\frac{z}{2\sqrt{(D^* t)}}\right) \tag{22}$$

which only depends on D^*. However, the important practical implication of equation (21) is that for real landfills where the source concentration varies with time, it is essential to know the values of D, ρK, n and not just the lumped parameter D^*. A technique for determining both D and ρK from a single test on an undisturbed sample of soil has been outlined by Rowe *et al.*[14] and described in detail by Rowe and Caers.[15]

The fact that the lumped parameter $D^* = D/(1 + \rho K/n)$ does not provide a good basis for predicting contaminant migration in landfills where the concentration varies with time can also be readily demonstrated by evaluating (21) at the surface of the deposit ($z = 0$) to give

$$c_T(t) = c(0,t) = \omega_0 \exp\left(\frac{n^2(1+\rho K/n)Dt}{H_f^2}\right)\operatorname{erfc}\left(\sqrt{\frac{n^2(1+\rho K/n)Dt}{H_f^2}}\right) \tag{23a}$$

At this surface location it is clear that the concentration of contaminant depends on the product $(1 + \rho K/n)D$ and not on the lumped parameter $D^* = D/(1 + \rho K/n)$. It is also evident from (23a) that the variation in surface concentration c_T with time can be expressed in terms of a unique function of dimensionless time T, viz.

$$c_T(T) = \omega_0 \exp(T) \operatorname{erfc}(\sqrt{T}) \tag{23b}$$

where

$$T = n^2(1 + \rho K/n)Dt/H_f^2 \tag{23c}$$

This variation in leachate concentration with dimensionless time is shown in Figure 3.

Variable mass input

In the foregoing examples, it was assumed that the mass of contaminant within the landfill reaches its peak value very quickly. However, if the time-dependent variation in mass input to the landfill is known, the effect of this can be modelled by splitting the time-dependent input into a series of linear portions and then using the principle of superposition together with equation (20). For example, if the mass input is assumed to vary linearly with time at a rate m up to a time t_0 after which no more mass is added, then the concentration at any depth z and time t is given by

$$c(z,t) = m\chi(t) \qquad \text{for } t < t_0 \tag{24a}$$

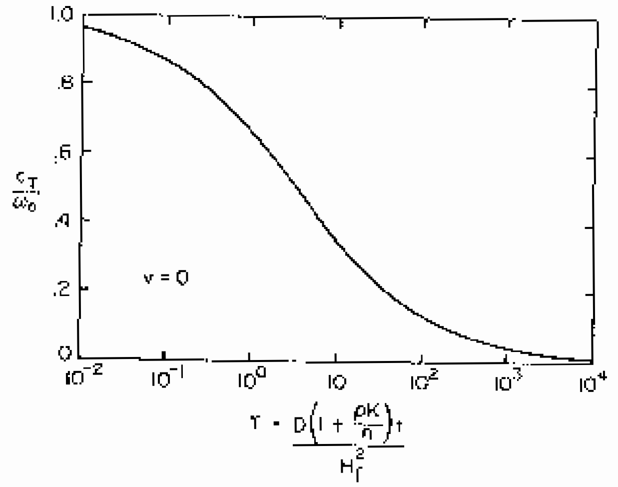

Figure 3. Variation in surface concentration c_T with dimensionless time

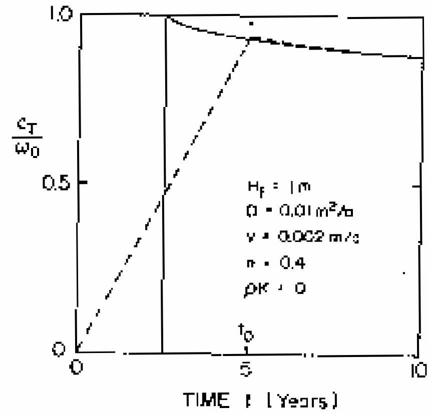

Figure 4. Effect of rate of mass input upon surface concentration variation with time

$$c(z,t) = m(\chi(t) - \chi(t-t_0)) \quad \text{for } t \geqslant t_0 \tag{24b}$$

where the function χ is defined in (20b).

To illustrate the effect of modelling the rate of increase in mass, the dashed line in Figure 4 shows the variation in surface concentration with time for the case where the mass input increases linearly with time for $t_0 = 5$ years and then no more mass is added. Note that if there were no mass transport into the soil, the maximum concentration at time $t = 5$ years would be ω_0. In fact, mass transport into the soil does occur reducing the maximum concentration to approximately $0{\cdot}93\,\omega_0$.

Although the increase in mass with time can be modelled using (20), an alternative approximate approach would be to assume that all the mass was deposited instantaneously at a time $0{\cdot}5\,t_0$ and then use (15). The results of an analysis is performed using this approximation are shown by the solid line in Figure 4. It will be seen that even at time t_0 there is very little difference between the surface concentrations. At a time $t = 2t_0$ (i.e. 10 years), the difference between the two solutions is less than 0·1 per cent. Although this is only an isolated example, calculations performed for many practical situations indicate that for time $t > 2t_0$, the approximate approach of assuming the landfill is constructed instantaneously at a time $0{\cdot}5\,t_0$ is accurate to better than 1 per cent.

CONCLUSIONS

Two analytic closed form solutions for the advection-dispersion equation have been presented. These solutions take account of changes in surface concentration with time as mass is transported into the soil. Consideration has also been given to time-dependent mass input into the landfill. The use of these solutions has been illustrated by a number of examples and it has been shown that consideration of the finite mass of contaminant within the landfill can significantly affect the predicted concentration profiles beneath the landfill.

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