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Postbuckling of Buried Flexible Tubes

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SUMMARY: The postbuckling behaviour of circular elastically-supported tubes is examined, and an approximate analytic solution to the problem is developed. The simplified analysis is shown to be an efficient and effective alternative to more costly numerical analysis.

1. INTRODUCTION

It has been recognised for some years that when long metal tubes are buried, the weight of the overlying soil generates significant hoop compressions which may act to destabilise the flexible structure. Many workers have sought to determine the strength of these soil-structure systems by solving the classical stability problem (e.g. Forrestal and Herrmann (1965), Cheney (1976), Moore and Booker (1983). The critical stress level so obtained does not, however, provide a complete picture of cylinder strength. A more complete analysis of both the prebuckling and post-buckling response is required if the likelihood of excessive deflection and structural yield, as well as the importance of postbuckling strength can be determined.

Recently the author has obtained simplified analytic solutions for the critical distributions of hoop compressions which act to destabilise elastically supported tubes, Moore and Booker (1983). Both smooth and rough interface behaviour was considered for tubes destabilised by uniform distributions of hoop compression. Geometrically non-linear analysis using the finite element method has also been undertaken, Moore and Booker (1984), and the postbuckling response of flexible elastic tubes has been examined in detail.

As a result of this work, the fundamental nature of tube postbuckling has been clarified, and the linearised analytic solutions are now extended so as to predict both the prebuckling and the post-buckling response of elastically supported tubes. The validity of the approximate analytic solution is then established on comparison with the more extensive numerical solution.

It is demonstrated that the simplified analytic postbuckling solution can be used to satisfactorily predict the response of elastically supported tubes at load levels up to twice that which is critical. Both the deformations and the bending moment distribution can be determined, so that this approximate analytic solution to the elastically supported tube problem is an efficient and effective alternative to more costly numerical analysis.

2. STATEMENT OF PROBLEM

The circular tube of radius a, uniform thickness t, Young's modulus E_ℓ and Poisson's ratio ν_ℓ, Figure 1, is assumed to be very long, so that it deforms under conditions of plane strain. The tube is assumed to be thin ($t/a << 1$) so that the membrane extensions are negligible and a simplified structural theory can be employed, Moore and Booker (1983).

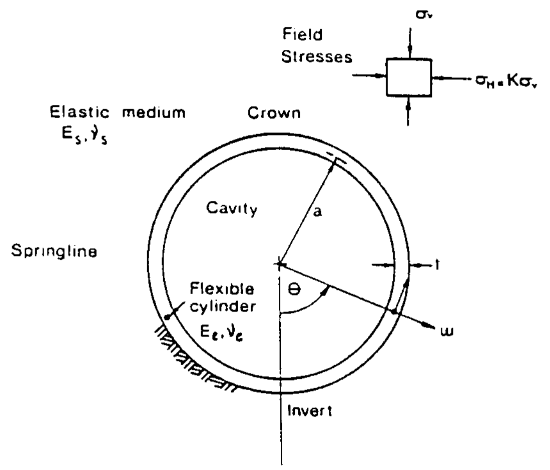

FIGURE 1: Co-ordinate Description

The supporting medium is assumed to be a single phase isotropic material such as a soil or rock mass, with an incrementally elastic behaviour characterized by two constants: Young's modulus E_s and Poisson's ratio ν_s.

Before insertion of the tube the continuum is assumed to be prestressed with uniform vertical σ_v and horizontal $\sigma_H = K\sigma_v$ field stresses, which induce the uniform hoop force N_0 (compression positive), in the tube.

Two alternative conditions will be assumed to characterise the continuum-structure interaction at the interface.

(a) A perfectly rough condition, resulting in complete compatibility of radial and circumferential displacements, and full transmission of normal and shear tractions across the interface.

(b) A perfectly smooth condition, where shear stress is not transmitted between the structure and ground, and where circumferential displacements are not continuous, due to interfacial slip.

The real interface condition for a buried tube will be somewhere between these two extremes since there will in general be some finite limit to the shear stresses that can be developed between ground and structure.

3. LINEAR RESPONSE

For this circular tube supported by an elastic continuum, both the structure and the continuum contribute towards the resistance of the system to any applied loads. In this section details are provided of existing linear analytic solutions which can be used to determine the tube response at load levels below that which is critical.

If the normal and circumferential displacements of the tube midsurface are w and v respectively, Figure 1, then it proves convenient to subdivide them into harmonic components

$$w = W_0 + \sum_{n=2}^{\infty} W_n \cos n\theta$$
$$v = \sum_{n=2}^{\infty} V_n \sin n\theta \qquad (1)$$

(the terms V_0, W_1 and V_1 have not been included since they describe the rigid body motion of the circular cylinder).

The uniform radial response of the buried tube W_0, is largely unaffected by any initial stresses in the structure. A simple static analysis of the continuum-structure system (e.g. Einstein and Schwartz (1979)) can thus be employed to find an expression for the uniform contraction W_0 in terms of the uniform component of the applied traction at the interface

$$W_0 = \frac{\sigma_v \ (1+K)\ a}{2(H + 2G_s)} \qquad (2)$$

where

$$H = E_\ell t/(1 - \nu_\ell^2) \qquad (3a)$$

is the hoop stiffness of the tube and

$$G_s = E_s/[2(1 + \nu_s)] \qquad (3b)$$

is the shear modulus of the elastic continuum.

The nonuniform deformation of the tube, represented by the terms (W_2, V_2, W_3, V_3 ...), is however influenced by the hoop force N, and a more complex analysis which takes account of the destabilising effect of the initial hoop compressions is required. Such a theory was developed by Moore and Booker (1983) for the analysis of tubes under the influence of uniform hoop compressions, and that solution is now described in detail.

Assuming that membrance extensions are negligible (Moore and Booker (1983))

$$\text{i.e.}\quad \varepsilon_{\theta\theta} = \frac{1}{a}\left(\frac{\partial v}{\partial\theta} + w\right) \cong 0 \qquad (4)$$

the linearised equations of equilibrium for a buried tube of flexural rigidity

$$D = E_\ell t^3/[12(1 - \nu_\ell^2)] \qquad (5)$$

solved for the harmonic coefficients of radial displacements W_n, are

$$W_n = \frac{n^2}{(n^2-1)^2}\ \frac{(\sigma_n - \tau_n/n)}{(n^2 D/a^4 - N_0/a^2 + B^n)} \qquad (6a)$$

if the load behaviour is constant directional or

$$W_n = \frac{n^2}{(n^2-1)^2}\ \frac{(\sigma_n - \tau_n/n)}{(n^2 D/a^4 - (N_0/a^2)\ [n^2/(n^2-1)] + B^n)} \qquad (6b)$$

if the load behaviour is hydrostatic. Prebuckling deformations occur as a result of the normal σ and tangential τ tractions applied at the interface, where these have the harmonic decompositions

$$\sigma = \sum_{n=2}^{\infty} \sigma_n \cos n\theta$$
$$\tau = \sum_{n=2}^{\infty} \tau_n \sin n\theta \qquad (7)$$

The continuum contributes towards the static stiffness of the tube through the elastic restraint coefficient B^n which takes the value

$$B^n = 2G_s a\ \frac{2n(1-\nu_s)-(1-2\nu_s)}{(n^2-1)(3-4\nu_s)} \qquad (8)$$

when the cylinder is rough and

$$B^n = 2G_s a\ \frac{n^2}{(n^2-1)\ [2n(1-\nu_s) + 1 - 2\nu_s]} \qquad (9)$$

for smooth cylinders.

Once the normal deformation coefficients W_n have been determined the zero extension condition (4) is used to obtain the circumferential deformation coefficients

$$V_n = -W_n/n \qquad (10)$$

The non-uniform interface tractions σ and τ act to disturb the buried tube from its initial undeformed position, and two sources of such disturbances will be considered in this work (Moore and Booker (1983)). Firstly, the presence of non-hydrostatic field stresses σ_v and $\sigma_H = K\sigma_v$ induces elliptical tractions with coefficients

$$\sigma_2 = \sigma_v(1 - K)/2$$
$$\tau_2 = \sigma_v(K - 1)/2 \qquad (11)$$

Any initial geometrical imperfections in the circular shape of the tube will also disturb the equilibrium of the system. The response of a tube with initial radius

$$r = a + \varepsilon \qquad (12)$$

such that

$$\varepsilon = \sum_{n=2}^{\infty} \varepsilon_n \cos n\theta \qquad (13)$$

and

$$\varepsilon_n = \frac{1}{\pi}\int_0^{2\pi} \varepsilon \cos n\theta\ d\theta \qquad (14)$$

can be conveniently found using the theory which has been presented for perfectly circular tubes, in conjunction with the disturbing tractions

$$\Delta\sigma_n = N_0\ \varepsilon_n(n^2 - 1)/a^2 \qquad (15a)$$

$\Delta\tau_n = 0$ (15b)

Stress resultants in the ring can also be found once the deformed shape has been determined. In particular, the bending moment M can be expressed as a function of the deformation

$$M = -D \sum_{n=2}^{\infty} (n^2-1) W_n \cos n\theta \qquad (16)$$

4. EXAMINATION OF TUBE POSTBUCKLING

The linear continuum-structure interaction theory which has been summarised in the previous section can be used to determine the response of flexible cylinders supported by an elastic medium. As the critical load level is approached, however, the deflections predicted become indeterminate, since the denominator of (6) becomes zero. This occurs because nonlinear deformation terms have been neglected in the differential equations of equilibrium for the circular tube. The redistribution of stress resultants associated with the critical deformation has also been neglected in the formulation. As a result the theory only performs adequately for load levels below that which is critical, and it fails to provide a solution in the postbuckling region.

More complete nonlinear analysis (using the numerical method described by Moore and Booker (1984)) provides a solution for the elastically supported tube problem in both the prebuckling and the postbuckling regions. A number of solutions to such problems will now be considered in greater detail.

A rough slightly imperfect tube

$(r = a(1 + 0.001 \cos 8\theta))$

is supported by an elastic medium with relative stiffness $D/E_s a^3 = 4.579 \times 10^{-4}$, Poisson's ratio $\nu_s = 0.3$, and is uniformly loaded with a normal traction which behaves hydrostatically. Eight critical waves develop around the cylinder, Figure 2, and the radial deformation at various points around one half-wavelength of the critical deformation is shown in Figure 3.

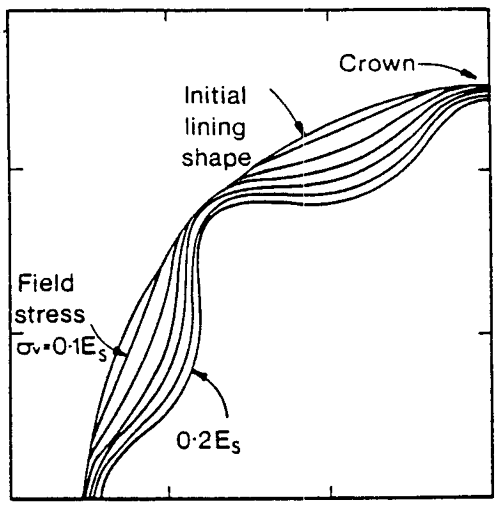

FIGURE 2: Deformation of Circular Tube with Imperfection

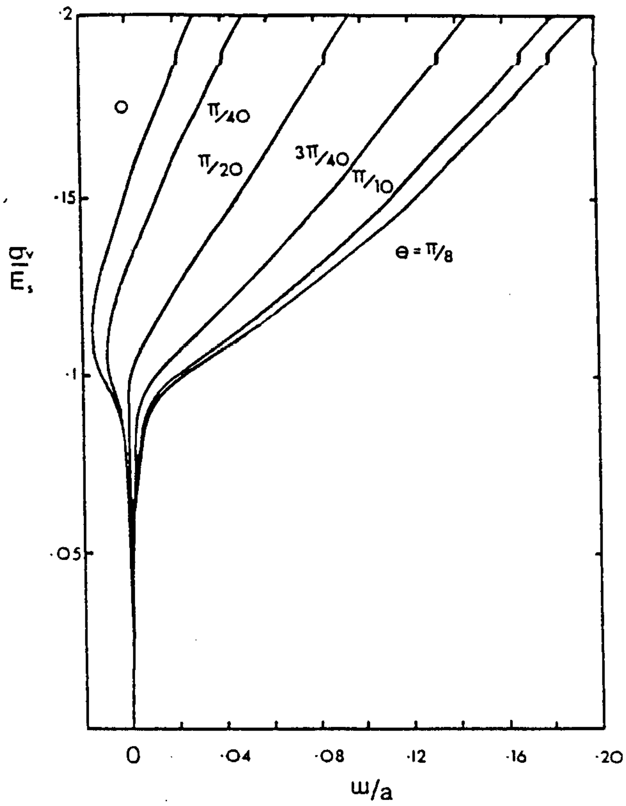

FIGURE 3: Radial Deformation of Elastically Supported Tube $D/E_s a^3 = 4.575 \times 10^{-4}$.

The initial deformation of the tube is predominantly composed of the harmonic bending deflections associated with the initial geometrical imperfection. However, a purely harmonic deformation about the initial position of the structure must have associated with it a gradual increase in the circumferential length of the tube. Since the tube is stiff in hoop extension and the change in length is restricted to the hoop compression $2\,W_0$, a general contraction $\bar{W}_0$ of the tube into the cavity must occur in conjunction with the bending deformation. This uniform contraction is clearly seen in both Figures 2 and 3, and since it is resisted by the continuum surrounding the tube, considerable postbuckling strength is developed.

To further illustrate this aspect of tube behaviour the distribution of hoop compressions around the cylinder is shown in Figure 4. The uniform hoop compression steadily increases in the prebuckling range as the in-situ stresses are released onto the flexible cylinder, but as the critical loads are approached the excess uniform tractions are redistributed to the continuum, and the uniform component of hoop compression remains below the critical level. The significant harmonic components of hoop compression associated with the harmonic deformation then develops.

5. ANALYTIC MODEL FOR POSTBUCKLING

For the approximate analytical description of elastically supported tube postbuckling, the mechanism by which the continuum-structure system responds will be assumed to be the sum of two components:

(a) The linear continuum-structure interaction

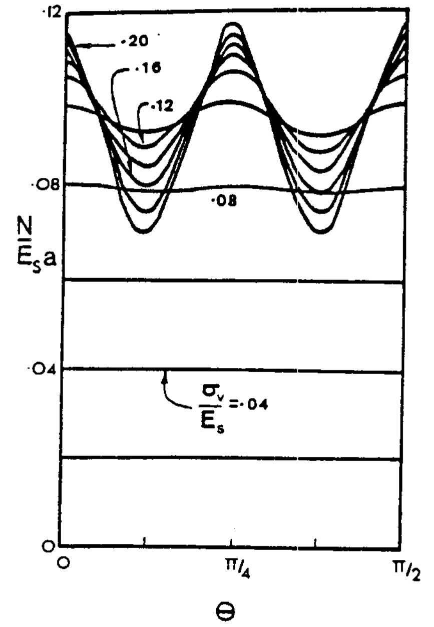

FIGURE 4: Hoop compressions in imperfect tube

response, as described by the theory outlined in section three.

(b) A nonlinear contraction of the structure into the cavity, as a result of the "linear" deformations and the high hoop stiffness of the tube.

The contraction will be assumed to occur so that the circumferential length of the tube

$$\ell = \int_0^{2\pi} (r^2 + (\partial r/\partial\theta)^2)^{\frac{1}{2}}\, d\theta \qquad (17a)$$

equals the initial length plus any hoop extension predicted by the linear theory

$$\ell = 2\pi(a + W_0) \qquad . \qquad (17b)$$

The radius of the deformed tube

$$r = a + W_0 + \bar{W}_0 + \sum_{n=2}^{\infty} (\varepsilon_n + W_n) \cos n\theta \qquad . \qquad (18)$$

The hydrostatic component of field stress

$$\sigma_m = (\sigma_V + \sigma_H)/2 \qquad (19)$$

is composed of two parts

$$\sigma_m = \sigma_m{}^a + \sigma_m{}^b \qquad (20)$$

where the superscripts a and b assign these components to the linear and nonlinear mechanisms respectively. The hydrostatic stress $\sigma_m{}^a$ is directly related to the linear coefficients W_0, W_2, V_2, W_3, V_3, ... through the equations of equilibrium and zero extension (6) and (16) respectively. The "nonlinear" component $\sigma_m{}^b$ is related to the nonlinear contraction $\bar{W}_0$ according to the resistance of the continuum to uniform deformation.

$$\bar{W}_0 = \frac{\sigma_m{}^b a}{2G_s} \qquad (21)$$

The circumferential length condition (17) provides a relationship between the linear and nonlinear components.

6. PERFORMANCE OF APPROXIMATE SOLUTION

A simplified postbuckling theory has been developed for elastically supported tubes, and its validity now needs to be examined through comparisons with the more complete nonlinear numerical solution. Figures 5 to 9 show the results of the study, based on the uniform problem, which has already been introduced (Figures 2 to 4) and some additional problems involving stiffer ground. On the basis of this work, a number of conclusions have been drawn.

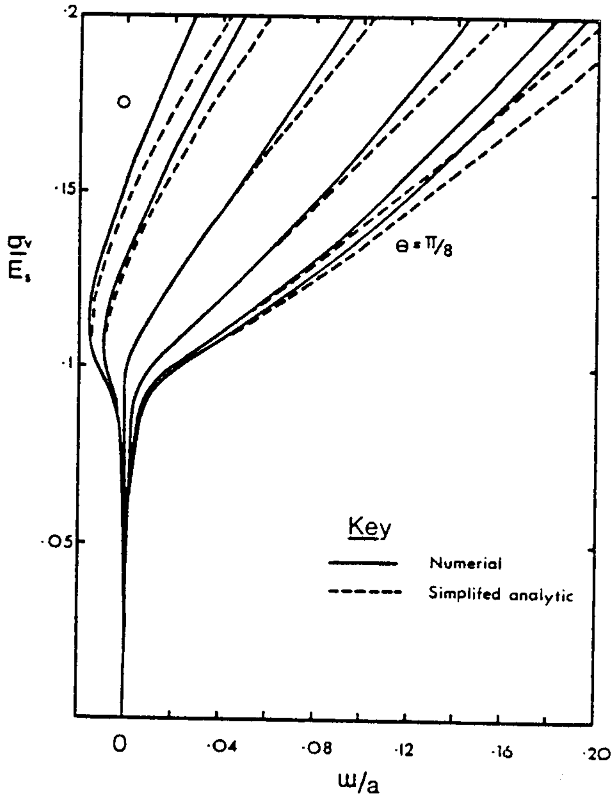

FIGURE 5: Radial Deformation of Elastically Supported Tube $D/E_s a^3 = 4.575 \times 10^{-4}$.

The simplified analytic solution can be used to satisfactorily determine the postbuckling response of the bonded elastic tube under uniform pressure, Figure 5. The numerical and analytic solutions are almost identical in the prebuckling and immediate postbuckling zones, but the simplified theory appears to underestimate the continuum resistance to the uniform "contraction" of the tube, so that deformations are overestimated at higher stress levels.

When the elastic medium is stiffer, the critical harmonic increases, so that the wavelength of the critical deformation becomes shorter. Figures 6 and 7 show the results for two such problems, where the critical harmonic n = 19 and 40 respectively. The responses shown in Figures 5 and 6 are very similar, but for the problem involving a very stiff continuum, Figure 7, the simplified analytic solution appears to slightly underestimate the structural deformations at high stress levels.

The simplified analytic theory can also be used to predict the bending moments which are generated in the uniformly loaded tube, and comparisons between numerical and analytic values are shown in Figures 8 and 9. The first figure shows the bending moment distribution around the tube at two load levels below that which is critical, and the two solutions match closely. Figure 9 shows a comparison at these higher stress levels, and it is apparent that the simplified theory does not perform quite as well in the postbuckling region. The numerical results indicate that the tube response is more complex than the simple harmonic bending assumed in the analytic model, as the length of the outward buckle tends to shorten. The magnitude of bending moment is then greater at the centre of that outward lobe than at the centre of the inward buckle. Nevertheless, a reasonable estimate of the maximum bending moment can still be made using the simplified theory.

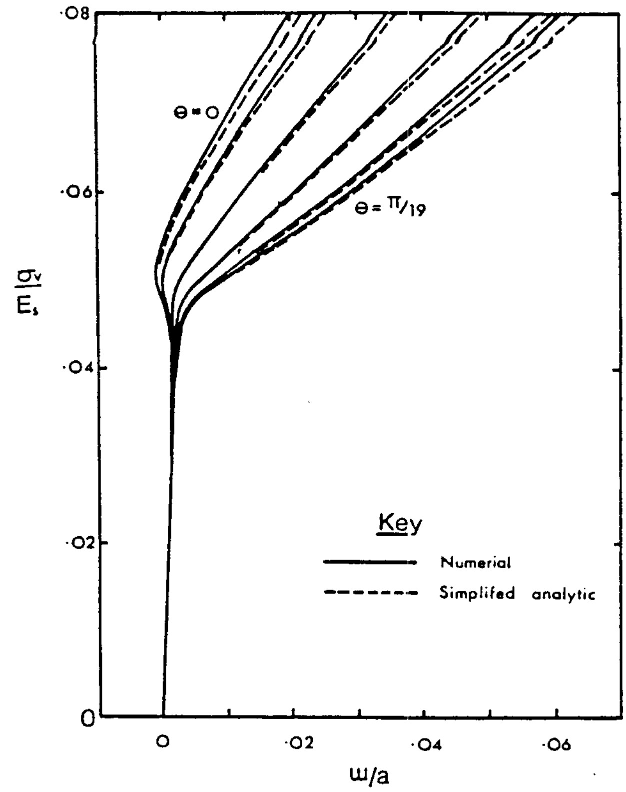

FIGURE 6: Radial Deformation of Elastically Supported Tube $D/E_sa^3 = 4.579 \times 10^{-5}$.

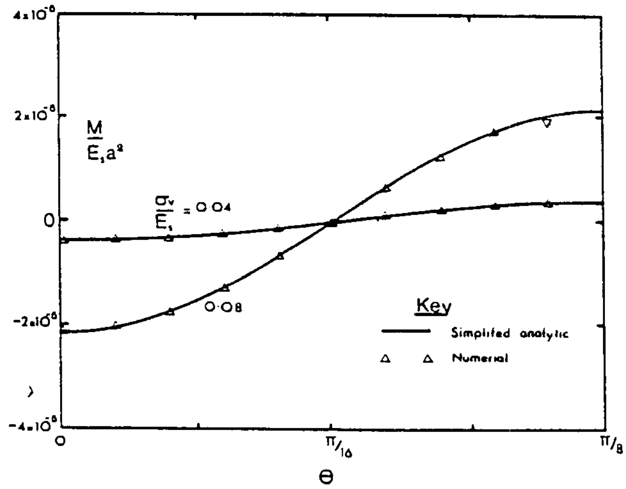

FIGURE 8: Bending Moments in an Imperfect Ring

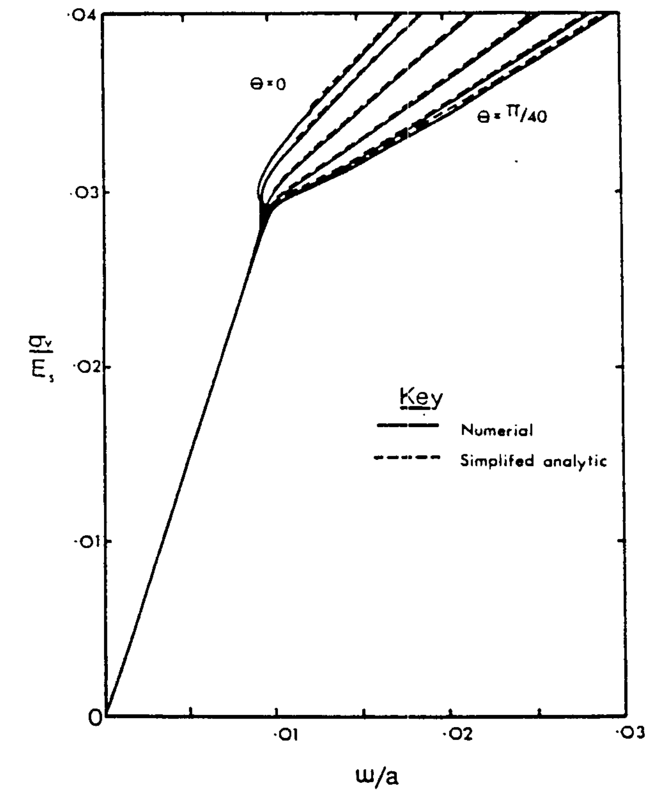

FIGURE 7: Radial Deformation of Elastically Supported Tube $D/E_sa^3 = 4.579 \times 10^{-6}$.

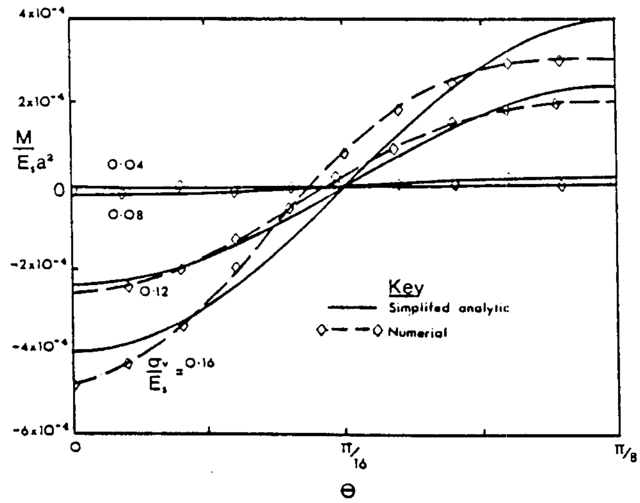

FIGURE 9: Bending Moments in an Imperfect Ring

7. CONCLUSIONS

The postbuckling response of circular elastically supported tubes has been examined, and a simplified analytic solution to the problem has been developed. The influence of uniform distributions of hoop compression has been considered, and as a result of a comparison between numerical and analytic solutions it can be concluded that:

(a) The prebuckling deformations and bending moments for an elastically supported tube can be determined very accurately using the simplified theory.

(b) The important elements of tube postbuckling have been incorporated in the approximate analytical solution, since the solution can be used to satisfactorily predict the response of elastically supported tubes up to about twice the critical load level.

(c) The solution will permit substantial savings in computational effort, by providing a reasonable alternative to more costly numerical solutions.

(d) The solution would facilitate studies into the postbuckling behaviour of elastically supported tubes, where the influence of ground stiffness, interface roughness and initial geometrical imperfections are investigated.

8. REFERENCES

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